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Vortex Rossby waves on smooth circular vortices Part I. Theory

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Abstract

A complete theory of the linear initial-value problem for Rossby waves on a class of smooth circular vortices in both *f*-plane and polar-region geometries is presented in the limit of small and large Rossby deformation radius. Although restricted to the interior region of barotropically stable circular vortices possessing a single extrema in tangential wind, the theory covers *all* azimuthal wavenumbers. The non-dimensional evolution equation for perturbation potential vorticity is shown to depend on only one parameter, *G*, involving the azimuthal wavenumber, the basic state radial potential vorticity gradient, the interior deformation radius, and the interior Rossby number.

In Hankel transform space the problem admits a Schrödinger's equation formulation which permits a qualitative and quantitative discussion of the interaction between vortex Rossby wave disturbances and the mean vortex. New conservation laws are developed which give exact time-evolving bounds for disturbance kinetic energy. Using results from the theory of Lie groups a nontrivial separation of variables can be achieved to obtain an exact solution for asymmetric balanced disturbances covering a wide range of geophysical vortex applications including tropical cyclone, polar vortex, and cyclone/anticyclone interiors in barotropic dynamics. The expansion for square summable potential vorticity comprises a discrete basis of radially propagating sheared vortex Rossby wave packets with nontrivial transient behavior. The solution representation is new, and for this class of swirling flows gives deeper physical insight into the dynamics of perturbed vortex interiors than the more traditional approach of Laplace transform or continuous-spectrum normal-mode representations. In general, initial disturbances are shown to excite two regions of wave activity. At the extrema of these barotropically stable vortices and for a certain range of wavenumbers, the Rossby wave dynamics are shown to become nonlinear for all initial conditions. Extensions of the theory are proposed.

Keywords: Continuous-spectrum; Cylindrical coordinates; Initial-value problem; Vortex Rossby waves

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1. Introduction

The tendency for a disturbed vortex to return to circular symmetry is called ‘vortex axisymmetrization’ (Melander et al., 1987). When it occurs, axisymmetrization nicely illustrates the upscale energy and downscale enstrophy cascade of two-dimensional turbulence (Kraichnan, 1967). Much work has been devoted to further understand this non-axisymmetric adjustment process in two-dimensional non-neutral plasma vortices (Briggs et al., 1970), two-dimensional Euler vortices (Bassom and Gilbert, 1998, 2000; Schecter et al., 2000; Balmforth et al., 2001; Bajer et al., 2001), two-dimensional oceanic and atmospheric vortices (McCalpin, 1987; Carr and Williams, 1989; Sutyrin, 1989; Guinn and Schubert, 1993; Ritchie and Holland, 1993; Smith and Montgomery, 1995; Montgomery and Kallenbach, 1997; Moller and Montgomery, 1999; Enagonio and Montgomery, 2001); and in three-dimensional tornado and tropical cyclone vortices (Nolan and Farrell, 1999; Montgomery and Enagonio, 1998; Moller and Montgomery, 2000; Reasor and Montgomery, 2001).

Recognizing the similarity between the shearing dynamics in a vortex and the shearing dynamics in unidirectional shear flow, many analytical and numerical studies have been devoted to understanding the dynamics of free transient disturbances in shear flows. Two lines of analytical investigation have been developed for this class of problems. The first follows the seminal works of Case (1960) and Dikii (1960) and is based on a singular ‘normal-mode’ expansion of transient disturbances in shear flows. In addition to the discrete normal modes, the expansion comprises a continuous-frequency-spectrum integral over the singular normal modes spanning all phase speeds which match the wind speed of the unidirectional basic state flow (e.g. Briggs et al., 1970; Held, 1985; Farrell, 1987; Balmforth and Morrison, 1999; Montgomery and Lu, 1977; Balmforth et al., 2000; Schecter et al., 2000). The second is based on so-called ‘non-modal’ solutions of the evolution equation and is an alternative way of representing sheared disturbances whose basis functions are not singular. The second approach has been less general than the first, since it has only been possible to apply it to simple and/or particular basic states. But it has still proven to be a fruitful source of results in the study of sheared vorticity disturbances (Thomson, 1887; Orr, 1907; Yamagata, 1976; Farrell, 1982, 1987; Boyd, 1983; Tung, 1983; Carr and Williams, 1989; Brunet, 1989; Sutyrin, 1989; Smith and Rosenbluth, 1990; Smith and Montgomery, 1995; Montgomery and Kallenbach, 1997).

It is known since Brown and Stewartson (1980) that regions of velocity extrema develop asymptotic behavior quite unlike that encountered with monotonic velocity profiles. The exact solutions obtained and studied for a parabolic jet by Brunet (1989) have shown how peculiar the linear dynamics of sheared disturbances can be at the extremum of a jet. Under certain conditions, such as a weak total vorticity gradient, these extremum can develop free coherent vortex structures by nonlinear processes for an arbitrarily small-amplitude initial condition as suggested by Brunet and Warn (1990) and demonstrated by Brunet and Haynes (1995).

This study will focus on localized transient vorticity disturbances on a class of smooth circular vortices with an angular velocity extremum at the origin. The approach will be of the second type. A discrete expansion of ‘non-modal’ solutions of the linearized potential vorticity (PV) equation is employed to completely solve the initial-value problem. We will show the relevance of this expansion by discussing different problems of geophysical fluid dynamic interest such as tropical cyclones, the polar vortex, and cyclone/anticyclone asymmetry in barotropic dynamics. In particular, the application of this new basis-set to the winter upper stratospheric polar vortex will explicitly show how qualitatively different the linear conservative dynamics are for the low azimuthal wavenumbers.

The outline of the paper is as follows. Section 2 derives the canonical model covering a wide range of geophysical vortex applications and then develops the complete linear theory of Rossby waves on smooth circular vortices with a single angular velocity extremum. Section 3 briefly considers linear aspects of the cyclone/anticyclone asymmetry problem. Section 4 discusses the relevance of the theory to tropical cyclone and polar vortex dynamics. Section 5 gives the conclusion.

2. Vortex Rossby waves in vortex interiors

2.1. A canonical disturbance equation for a class of circular vortices possessing an extremum in the tangential winds

2.1.1. Nondivergent vorticity dynamics

We begin the model development with the inviscid nondivergent vorticity equation in the plane. This model will be generalized in Section 2.1.2. The nondivergent vorticity equation is given by

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{\partial P}{\partial t} + \vec{V} \cdot \vec{\nabla} P = 0 \tag{2.1}$$

where $P = f + \zeta$ is the absolute vorticity, f the Coriolis parameter, ζ the relative vorticity, and $\mathrm{d}/\mathrm{d}t$ is the material derivative operator in the plane. A cylindrical-polar coordinate system is adopted with r the radius and λ the azimuthal angle. Because of the nondivergent assumption there exists a streamfunction such that

$$\zeta = \Delta\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\lambda^2}, \qquad u = -\frac{1}{r}\frac{\partial\psi}{\partial\lambda}, \qquad v = \frac{\partial\psi}{\partial r} \tag{2.2}$$

For small-amplitude monochromatic disturbances of wavenumber n on a circular basic state vortex, products of perturbations are neglected in the first approximation and the vorticity equation simplifies to

$$\frac{\partial \Delta_n\phi}{\partial t} + \mathrm{i}n\,\bar{\Omega}\,\Delta_n\phi - \mathrm{i}n\,\gamma\phi = 0 \tag{2.3}$$

where

$$\Delta_n\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) - \left(\frac{n^2}{r^2}\right)\phi \tag{2.4}$$

is the disturbance vorticity in azimuthal Fourier space, $\bar{\Omega} = (\bar{v}(r))/r$ is the basic state angular velocity, and γ is the radial absolute vorticity gradient (Rossby restoring mechanism) divided by radius

$$\gamma = \frac{1}{r}\frac{\mathrm{d}\bar{P}}{\mathrm{d}r} = \frac{1}{r}\left(\frac{\mathrm{d}f}{\mathrm{d}r} + \frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{1}{r}\frac{\mathrm{d}(r\bar{v})}{\mathrm{d}r}\right]\right) \tag{2.5}$$

Here the Coriolis parameter is assumed to be a function of radius only, the justification of which is provided below. The perturbation streamfunction in real space is recovered via $\psi = Re(\mathrm{e}^{\mathrm{i}n\lambda}\phi(r,t))$.

Throughout this work all fields (basic state and perturbations) are assumed finite at the origin. In particular, we assume that $\bar{v}_0 = \bar{v}(0) = 0$, thereby excluding point vortices as candidates for the mean vortex. Henceforth, our discussion will be limited to the study of *localized disturbances* around the origin of a smooth vortex and we will employ a Taylor series expansion near the origin for the basic state variables. We then obtain the following leading order terms for γ

$$\gamma = \left(\frac{\mathrm{d}^2 f}{\mathrm{d}r^2} + \frac{4}{3}\frac{\mathrm{d}^3\bar{v}}{\mathrm{d}r^3}\right)\bigg|_{r=0} + \frac{3}{2}\frac{\mathrm{d}^2\bar{v}}{\mathrm{d}r^2}\bigg|_{r=0}\frac{1}{r} \tag{2.6}$$

where $f_0' = f'(0)$ is assumed zero. The latter condition is relevant for two types of geophysical vortices. Tropical cyclones and the circumpolar vortex using an orthogonal polar projection near the pole. The relatively small spatial scale of a tropical cyclone justifies the use of the *f*-plane approximation to examine the linear dynamics of its near-core asymmetries (e.g. Carr and Williams, 1989; Montgomery and Kallenbach, 1997). The orthogonal polar projection of the sphere near the polar axis (hereafter called 'the *polar* tangent-plane approximation') shows that f_0' vanishes at the pole (Leblond, 1964). Recall that the polar tangent-plane approximation replaces the sine of colatitude by the identity function, that is $(\sin(\varphi) - \varphi)/\varphi \ll 1$. As an illustration of its accuracy for colatitude of 45 and 60°, the relative error is approximately 10 and 21%, respectively.

Now the third term in Eq. (2.6) leads to a potentially singular γ at the origin. This does not necessarily imply an unbounded term in the differential equation since bounded solutions to ϕ in Eq. (2.3) behave as r^n as $r \to 0$, rendering the product $\gamma\phi$ in Eq. (2.3) finite at the origin for all n. For simplicity, however, we will assume that $\mathrm{d}^2\bar{v}/\mathrm{d}^2r$ vanishes at $r = 0$. For an analytic basic state vorticity distribution that decreases monotonically with radius, the vorticity profile is equivalent to a Gaussian profile near the origin (Bassom and Gilbert, 1998; their Section 2). Further discussion of the geophysical vortices that can be studied with this constraint is given in Sections 2.1.2 and 4. The basic state angular velocity distribution corresponding to a vanishing $\mathrm{d}^2\bar{v}/\mathrm{d}^2r$ is then

$$\bar{\Omega} = \bar{\Omega}_0 + \bar{v}_0'''\frac{r^2}{3!}$$

where prime is shorthand for radial derivative. The constant term in the angular velocity can be eliminated in Eq. (2.3) by defining a modified streamfunction amplitude such that $\phi = \exp(-\mathrm{i}n\,\bar{\Omega}_0 t)\tilde{\phi}$. This is a reminder that in the nondivergent approximation, the linear (and nonlinear) vorticity dynamics are invariant under a transformation to a uniformly rotating frame of reference (cf. Melander et al., 1987).

2.1.2. Finite depth and cyclone/anticyclone asymmetry effects

Although the nondivergent model (2.3) is valid for arbitrary Rossby numbers, for it to be strictly applicable to atmospheric or oceanic vortices it must be interpreted as the limit of an infinitely deep vortex. The nondivergent model suffers from the additional defect that circular cyclonic and anticyclonic vortices with identical wind structure behave the same when similarly perturbed. Direct numerical simulations in finite-depth flows away from horizontal boundaries suggest, however, that for small but finite Rossby numbers, anticyclones are more robust than cyclones (Cushman-Roisin and Tang, 1990; Polvani et al., 1994; Yavneh et al., 1997; Stegner and Dritschel, 2000). To further understand this phenomenon in barotropic dynamics a simple balance model capturing the leading-order evolutionary differences between cyclones and anticyclones is desired.

For a vortex possessing a finite but subcritical Froude number subject to a small-amplitude near-core perturbation, Eq. (2.3) can be generalized to cover both finite-depth and cyclone/anticyclone asymmetry effects using the asymmetric balance (AB) theory of Shapiro and Montgomery (1993). If in the core region of the vortex the rotation rate and height are approximated as constant except when radially differentiated (a vortex 'beta-plane' approximation), the linearized AB equation in shallow water dynamics can be simplified to:

$$\left[\frac{\partial}{\partial t} + \mathrm{i}n\left(\frac{1}{3!}r^2\bar{v}_0'''\right)\right](\Delta_n\tilde{\phi} - L_\mathrm{R}^{-2}\tilde{\phi}) - \mathrm{i}n\,\gamma\tilde{\phi} = 0 \tag{2.7}$$

where

$$\gamma = f_0'' + \frac{4}{3}\bar{v}_0''' - \frac{\bar{\Omega}_0}{L_\mathrm{R}^2}\frac{f_0+\bar{\Omega}_0}{f_0+2\bar{\Omega}_0} \tag{2.8}$$

is the radial PV gradient (Rossby restoring mechanism) divided by radius in the polar tangent-plane approximation (cf. Eq. (2.6)); $\tilde{\phi}$ is a transformed geopotential amplitude; $L_\mathrm{R}^2 = \bar{\phi}_0/(f_0 + 2\bar{\Omega}_0)^2$ is the square of the local Rossby radius evaluated at the vortex center; and $(\Delta_n\tilde{\phi} - L_\mathrm{R}^{-2}\tilde{\phi})$ is proportional to the disturbance PV. Following Section 2.1.1, we have assumed that $\mathrm{d}^2\bar{v}/\mathrm{d}^2r$ vanishes at $r = 0$. Other notation is as in Section 2.1.1 (see Appendix A for a derivation of Eqs. (2.7) and (2.8)).

It is evident from Eqs. (2.7) and (2.8) that finite depth and cyclone/anticyclone asymmetry effects enter via L_R and γ. In tropical cyclones, the f_0'' term in Eq. (2.8) is absent. Tilde notation is henceforth dropped. Unlike the nondivergent model, the $\bar{\Omega}_0$ term remains both in the Rossby restoring term and the invertibility relation. The vorticity dynamics are thus no longer invariant under a transformation to a uniformly rotating frame of reference.

2.1.3. A canonical model for perturbed vortex interiors

A model encompassing both Eqs. (2.3) and (2.7) follows immediately. The change of variables

$$r = L_\mathrm{R}r', \qquad \gamma' = \frac{3!\gamma}{\bar{v}_0'''} = 8 + \frac{3![f''(0) - (\bar{\Omega}_0/L_\mathrm{R}^2)(f_0+\bar{\Omega}_0)/(f_0+2\bar{\Omega}_0)]}{\bar{v}_0'''} \tag{2.9}$$

and

$$t' = ntL_\mathrm{R}^2\frac{1}{3!}\bar{v}_0'''$$

yields the re-scaled equation for Eq. (2.7):

$$\mathrm{i}\frac{\partial q}{\partial t} = r^2 q - \gamma\phi \tag{2.10}$$

where

$$q = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) - \left(\frac{n^2}{r^2} + 1\right)\phi \tag{2.11}$$

is the disturbance PV. In Eq. (2.10), prime notation for dimensionless variables has been dropped. The nondivergent model of Section 2.1.1 is recovered on taking L_{R} to infinity, interpreting ϕ as streamfunction amplitude, and deleting the last term in the invertibility relation Eq. (2.11).

We now discuss the general mathematical properties of localized disturbances for the initial-value problem of Eq. (2.10). The definition of localness in this work is any solution to Eq. (2.10) with square summable disturbance PV and geopotential in the plane, that is L^2:

$$\int_0^\infty qq^* r\,\mathrm{d}r \le C \quad \text{and} \quad \int_0^\infty \phi\phi^* r\,\mathrm{d}r \le C'\ \forall t \tag{2.12}$$

where C and C' are finite numbers. Note that the weight function r takes into account the cylindrical geometry. Square summability means that the perturbation geopotential amplitude must vanish at infinity. The latter is necessary (but not sufficient) for geopotential perturbations to be members of L^2.

2.2. The linear evolution equation in transform space: a Schrödinger wave equation

The solution of the initial-value problem begins by applying a Hankel transform to Eq. (2.10). The Hankel transform $\boldsymbol{H}_n[g(r)]$ of an L^2 function g and its inverse are, see Davies (1985) or Ditkine and Proudnikov (1982):

$$\boldsymbol{H}_n[g](\kappa) = \int_0^\infty g(r)J_n(\kappa r)r\,\mathrm{d}r \tag{2.13}$$

and

$$g(r) = \int_0^\infty \boldsymbol{H}_n[g](\kappa)J_n(r\kappa)\kappa\,\mathrm{d}\kappa$$

From the properties of the Hankel transform, it follows that

$$\begin{aligned}\boldsymbol{H}_n[q](\kappa,t) &= -(\kappa^2+1)\boldsymbol{H}_n[\phi](\kappa,t),\\ \boldsymbol{H}_n[r^2 q](\kappa,t) &= -\frac{1}{\kappa}\frac{\partial}{\partial\kappa}\left(\kappa\frac{\partial\boldsymbol{H}_n[q]}{\partial\kappa}\right) + \frac{n^2}{\kappa^2}\boldsymbol{H}_n[q]\end{aligned} \tag{2.14}$$

Using Eqs (2.13) and (2.14), Eq. (2.10) may be transformed into the following equation:

$$\mathrm{i}\frac{\partial T}{\partial t} = -\frac{\partial^2 T}{\partial k^2} + \left[\frac{n^2 - 4^{-1}}{\kappa^2} + \frac{\gamma}{\kappa^2+1}\right]T \tag{2.15}$$

where

$$T(\kappa, t) = k^{1/2}\boldsymbol{H}_n[q](\kappa, t)$$

The vanishing of $J_n(r\kappa)$ at the origin for all $n \neq 0$ together with Parseval's theorem implies that $T(\kappa, t)$ must vanish at both $\kappa = 0$ and $\kappa = \infty$.

We recognize Eq. (2.15) as a one-dimensional Schrödinger's equation (in momentum space) with the potential

$$V(\kappa) = \frac{n^2 - 4^{-1}}{\kappa^2} + \frac{\gamma}{\kappa^2 + 1} \tag{2.16}$$

where the reader is referred to Landau and Lifshitz (1966) and Morse and Feshbach (1953) for a discussion of Schrödinger's equation. The 'normal modes' are solutions of Eq. (2.15) and are defined by

$$T_{n,j}(k)\exp(-\mathrm{i}\omega_{n,j}t) \tag{2.17}$$

where $\omega_{n,j}$ is the time frequency.

The existence of a discrete spectrum of vortex Rossby waves is possible when V is a potential well. A unique minimum for the potential is found at

$$\kappa_{\mathrm{m}} = \left[\sqrt{\frac{-\gamma}{n^2 - 4^{-1}}} - 1\right]^{-(1/2)} \tag{2.18}$$

i.e. when $-\gamma/(n^2 - 4^{-1}) > 1$. Using the results of Cohen-Tannoudji et al. (1977, Chapter 3) it can be shown that the frequency is bounded by the following inequality:

$$0 > \omega_{n,j} > \frac{-\gamma(n^2 - 4^{-1})}{(\sqrt{-\gamma} - \sqrt{n^2 - 4^{-1}})^2} \tag{2.19}$$

In Appendix B an explicit frequency-spectrum formula for different approximations is provided. These formulas indicate that at the first fundamental level (i.e. $j = 0$) a very small frequency should appear for $n = 1$ in the vicinity of $\gamma = -4$ (see Section 3c of Montgomery and Brunet (2002) for an example).

The potential V is a potential barrier when $(-\gamma/(n^2 - 4^{-1})) \leq 1$ and it is well known that there is no discrete spectrum solution for Eq. (2.15) in this case, but there is a continuous-spectrum. The study of the continuous-spectrum is generally a difficult problem. To gain further insight into the dynamics of Eq. (2.15) two levels of approximation will be assumed in order to simplify the analysis: a large or small radius of deformation relative to the radial scale of the disturbance, i.e. $\kappa \gg 1$ and $\kappa \ll 1$, respectively. The corresponding 'potentials' obtained are

$$V(\kappa) = \frac{n^2 - 4^{-1} + \gamma}{\kappa^2}, \quad \text{for } \kappa \gg 1 \tag{2.20}$$

and

$$V(\kappa) = \frac{n^2 - 4^{-1}}{\kappa^2} + \gamma, \quad \text{for } \kappa \ll 1$$

A matching with an inner solution for $\kappa \gg 1$ and an outer solution for $\kappa \ll 1$ to produce a uniformly valid solution to the exact equation will be discussed in future work. The approximation with the small radius of deformation can be 'Doppler-shifted' to eliminate the constant γ (as was done in Sections 2.1.1 and 2.1.2 to eliminate the $\bar{\Omega}_0$ term in azimuthal advection). We thus limit ourselves to two limiting cases:

$$V(\kappa) = \frac{G}{\kappa^2} \tag{2.21}$$

where

$$G = \gamma + n^2 - 4^{-1}, \quad \text{for } \kappa \gg 1, \qquad G = n^2 - 4^{-1}, \quad \text{for } \kappa \ll 1 \tag{2.22}$$

The case with large radial scale is essentially equivalent to the one with small radial scale with $\gamma = 0$, but the approximate inversion formula for the geopotential is different. The disturbance geopotential in the former case is then proportional to the disturbance PV and this should be remembered when interpreting the upcoming results. Here we use the large deformation radius approximation keeping in mind that the introduction of a small deformation radius is equivalent to taking null γ with the disturbance geopotential proportional to disturbance PV. These two types of approximations assure us of the robustness of our results in the intermediate case of a finite deformation radius. A future study will consider a detailed analysis of the small deformation radius limit.

The Schrödinger's equation potential Eq. (2.21) was also obtained by Brunet (1989) in a similar fashion for the problem of propagating Rossby waves on a zonal jet flow near a wind extremum. In Brunet's study, the spectral transform employed was a Fourier transform in the Cartesian coordinate y of the β-plane approximation. But unlike the zonal-jet problem where the analytical results were only asymptotically valid in the limit of a vanishing zonal wavenumber, the analytical results obtained here are exact for all azimuthal wavenumbers.

In the following sections it will be shown that the initial-value problem can be solved for all time for localized disturbances evolving according to Eq. (2.15) with the potential (2.21) when $G \geq 3/4$. Note that T is square summable, if $(T, T) \leqslant C \ \forall t$, where $(A, B) = \int_0^\infty AB^* \, \mathrm{d}k$ is the canonical scalar product in transform space.

2.3. Solving the initial-value problem

Miller (1977; Section 2.3; Eq. (3.14)) in applying Lie group analysis to separation-of-variable problems introduced a complete discrete orthonormal basis of L^2 functions that solved the initial-value problem of Eq. (2.15) for all time when $G \geq 3/4$. The basis is not the usual 'normal-mode' eigenfunction expansion associated with time-space translations, but is related to the space-time dilatation symmetry of Eq. (2.15) with the potential Eq. (2.21). The expansion is

$$T(k, t) = \sum_{p=0}^{\infty} a_p T_p(k, t)$$

where

$$a_p = (T(k, 0), T_p(k, 0))$$

and

$$T_p(k,t) = (-2)^p \exp\left[\mathrm{i}\pi\,\frac{\mu+2}{4}\right](t-\mathrm{i})^{-((\mu+3)/4)-p}(t+\mathrm{i})^{((\mu+1)/4)+p} \times\left(\frac{\kappa^2}{1+t^2}\right)^{(\mu+1)/4}\exp\left[\frac{\kappa^2(\mathrm{i}t-1)}{4(1+t^2)}\right]L_p^{\mu/2}\left(\frac{1}{2}\frac{\kappa^2}{1+t^2}\right) \tag{2.23}$$

where

$$\mu = \sqrt{4G+1} = 2\sqrt{\gamma+n^2} \tag{2.24}$$

and $L_p^{(\alpha)}(z)$ is a Laguerre polynomial of order p with $L_p^{(\alpha)}(0) = 1$ (see for example, Abramowitz and Stegun, 1970, chapter 22). The proof of the completeness in L^2 space is valid for $\mu \geq 2$ or $G \geq 3/4$. The latter condition ensures that Eq. (2.15) is self-adjoint and this makes the geopotential transform an L^2 function. Note that Parseval's equality ensures that L^2 disturbances in transform space are also L^2 in real space. We have thus obtained for all time a complete solution of the initial-value problem (Eq. (2.10)) with square summable disturbance geopotential, velocity, and PV.

It is worth noting that for a normal-mode continuous-spectrum problem, we have surprisingly obtained a discrete expansion of radially propagating wave packets each of which (upon inversion to physical space) is a smooth function of radius. This is certainly an interesting alternative to the more classical approach of Case (1960) and Dikii (1960) using Green functions and Laplace transforms, and we believe is a more fundamental route to understanding the asymmetric dynamics of vortex interiors for this class of vortices. The discrete nature of the Rossby wave dynamics indicated by the above series is believed fundamental and not an artifact of the particular potential Eq. (2.21) employed. Evidence in support of this belief is found in the numerous linear initial-value simulations reported by Montgomery and Kallenbach (1997), Moller and Montgomery (1999) and Bassom and Gilbert (1999) using more realistic mean state vortices whose relative vorticity decreases to zero at large radius (e.g. Figs. 2–5, 9c and 11 of Montgomery and Kallenbach, 1997; Figs. 2, 3 and 7 of Moller and Montgomery, 1999; and Figs. 2, 6 and 7 of Bassom and Gilbert, 1999).

The case $G < 3/4$ is pathological because the basis functions Eq. (2.23) are too singular at the origin in momentum space, that is $|T_p/k^2|^2 \sim \mathrm{O}(\kappa^\alpha)$, where $\alpha < -1$, and makes the disturbance geopotential not normalizable. This is due to the collapse of the disturbance geopotential on the largest scale. Note that the disturbance PV stays square summable for all G. Using results of Szegö (1959, p. 108), one can in fact show that the function Eq. (2.23) forms a complete L^2 basis for the disturbance PV at $t = 0$ and consequently for all time, if $G \geq -(1/4)$. The presence of a possible discrete spectrum when $G < -(1/4)$ forbids any a priori proof of completeness for the basis Eq. (2.23), but clearly this basis is square summable for all G. The difficulty then is to obtain a square summable geopotential for the continuous-spectrum. This problem can be overcome by not neglecting the radius of deformation effect in the inner region when obtaining the geopotential from the PV, that is, use the exact inversion formula given at Eq. (2.14). The resulting geopotential will then be square summable, but will have a slow radial decay in physical space for $G < 3/4$.

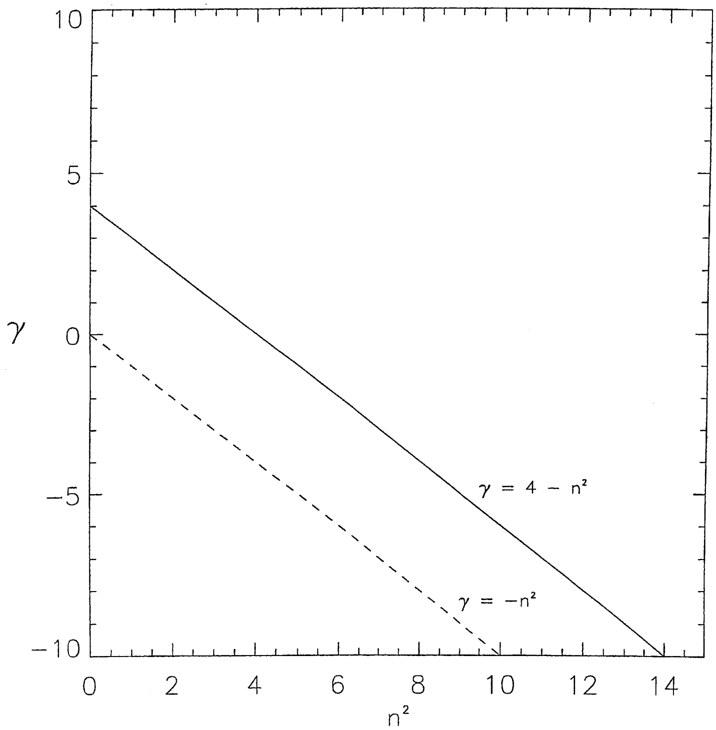

Fig. 1. Parameter space for the vortex model Eq. (2.10). On the abscissa is n^2 and on the ordinate γ. The region lying above the solid line corresponds to $G \geq 3/4$, while the region lying below the dashed line correspond to $G < -(1/4)$ (see text for further details).

In summary, the expansion Eq. (2.23) represents a uniformly valid square summable expansion for the disturbance PV satisfying Eq. (2.15) with the potential Eq. (2.21) for all G and all time. The disturbance geopotential is uniquely defined by the inversion formula (Eq. (2.14)) and is square summable, but not localized inside a region circumscribed by the radius of deformation when $G < 3/4$. The completeness of the expansion can only be guaranteed for $G \geq -(1/4)$, due to the possible existence of a discrete spectrum when $G < -(1/4)$. In terms of the parameters γ and n^2, the condition permitting the existence of discrete modes ($G < -(1/4)$) is found below the dashed line in Fig. 1. An example of a discrete vortex Rossby wave in the latter case is presented in Montgomery and Brunet (2002).

2.4. Asymptotic behavior of an arbitrary initial condition in real space

Eq. (2.23) shows that all initial conditions asymptotically, $t \gg 1$, converge to an expression independent of p, except for the order of the Laguerre polynomials. The Gaussian factor ensures that the Laguerre polynomials are localized in κ^2/t^2, which makes asymptotically the small radial scales different for each polynomial, but for all p none of them will dominate for $\kappa^2/t^2 \geq O(1)$. The constant term in the Laguerre polynomials will dominate when $\kappa \leq \mathrm{O}(1)$, which makes the asymptotic (large t) behavior of any initial conditions characterized by the term $p = 0$ in Eq. (2.23). For the normalized basis with $p = 0$, we can invert the Hankel transform, Eqs. (2.13) and (2.14), to obtain the dominant Fourier PV and geopotential amplitude in radius-time space (Gradshteyn and Ryzhik, 1980, Eq. (6.631)):

$$\hat{q}(r,t) = (2n+\mu)D(1+\mathrm{i}t)^{-(\mu/4)}[-r^2(1+\mathrm{i}t)]^{n/2} \times M\left[\frac{n}{2}+\frac{\mu}{4}+1, n+1, -r^2(1+\mathrm{i}t)\right]$$

and

$$\hat{\phi}(r,t) = D(1+\mathrm{i}t)^{-(\mu/4)-1}[-r^2(1+\mathrm{i}t)]^{n/2} M\left[\frac{n}{2}+\frac{\mu}{4}, n+1, -r^2(1+\mathrm{i}t)\right]$$

where

$$D = a_0 \exp\left[\mathrm{i}\pi\,\frac{2\mu+2n+4}{4}\right]\frac{2^{(\mu/2)-1}}{\Gamma(n+1)}\Gamma\left(\frac{n}{2}+\frac{\mu}{4}\right) \tag{2.25}$$

and $M(a, b, z)$ is Kummer's function. A solution similar to (2.25) was obtained by Bassom and Gilbert (1998; their Eq. (6.16)) for Gaussian-type basic state vortices. Bassom and Gilbert (1998) obtained their solution in the planar nondivergent case by employing a large-time asymptotic expansion. The complete orthonormal basis Eq. (2.23) of vortex Rossby wave-packets was not obtained by Bassom and Gilbert (1998) and is new. The leading solution Eq. (2.25) is to be compared with analogous solutions found in other shear flow problems by Thomson (1887), Orr (1907), Yamagata (1976), Boyd (1983), Tung (1983), Farrell (1987), Brunet (1989), Carr and Williams (1989), Smith and Rosenbluth (1990), Smith and Montgomery (1995), and Montgomery and Kallenbach (1997). From the form of Eq. (2.25) we see that there are two asymptotic ($t \gg 1$) regions in real space: an inner region $|r^2(1+\mathrm{i}t)| \sim \mathrm{O}(1)$ and an outer region $|r^2(1+\mathrm{i}t)| \gg 1$.

The inner-region solutions are obtained from Eq. (2.25) directly, showing the following decay for the disturbance PV and geopotential amplitudes:

$$\hat{q}(r,t) \sim \mathrm{O}(t^{-(\mu/4)}) \quad \text{and} \quad \hat{\phi}(r,t) \sim \mathrm{O}(t^{-(\mu/4)-1}) \tag{2.26}$$

Note that the geopotential decays faster than PV by a factor t^{-1}. The temporal decay for PV and geopotential is generally unlike the familiar one-over-t-squared decay of stream-function amplitude and constant vorticity amplitude found with sheared disturbances in rectilinear unbounded Couette flow (Thomson, 1887; Orr, 1907; Farrell, 1987) and circular Couette flow (Carr and Williams, 1989; Smith and Montgomery, 1995). Near the origin the disturbance PV amplitude for wavenumber n varies like r^n. Since $r^2 t = \mathrm{O}(1)$, the PV decay given by Eq. (2.26) coincides with that of Couette flow only when the azimuthal wavenumber becomes large, i.e. $n^2 \gg \gamma$. Because of the vanishing differential rotation at the origin, geopotential decay in the inner region is generally slower than in rectilinear Couette flow (Section 3b gives an exception). For low azimuthal wavenumbers, PV disturbances generally behave quite differently from a passive scalar (for a phenomenological explanation, see for example, Sections 2.4–2.6 of Montgomery and Kallenbach, 1997; Bassom and Gilbert, 1998). The solution Eq. (2.23) clarifies the nature of the temporal decay in the inner region that was not captured with the local WKB analysis of Montgomery and Kallenbach (1997).

Substitution of the asymptotic solutions into Eq. (2.1) shows that the small amplitude expansion remains uniformly valid, i.e. the nonlinear advection terms are asymptotically smaller than the linear terms, at the next order in the inner region for $\mu \geq 4$, while for $\mu < 4$ the amplitude expansion become disordered after $t \sim \mathrm{O}(\varepsilon^{4/(\mu-4)})$, where ε is the small-amplitude expansion parameter. Using Eq. (2.24) we conclude that disturbances with $4 > \gamma + n^2$ can potentially bring an inconsistency in the linear approximation. Note that the dimensional time for the amplitude of the nonlinear terms to become comparable to the amplitude of the linear terms is dependent on the azimuthal wavenumber. The ratio τ between the 'breaking time' for $n = 1$ and other azimuthal wavenumbers is, using Eq. (2.9)

$$\tau = \begin{cases} \infty & \text{if } \gamma + n^2 \geq 4 \\ \dfrac{1}{n}\varepsilon^{(2/(\sqrt{\gamma+n^2}-2))-(2/(Re(\sqrt{\gamma+1})-2))} & \text{if } 4 > \gamma + n^2 \geq 0 \\ \dfrac{1}{n} & \text{if } 0 > \gamma + n^2 \end{cases} \tag{2.27}$$

Note that the breaking time for $n = 1$ is proportional to $\varepsilon^{2/(Re(\sqrt{\gamma+1})-2)}$ and then Eq. (2.27) indicates that the azimuthal wave number 1 expansion will always get disordered before other wavenumbers n, if $3 > \gamma > -n^2$.

A similar disordered expansion scenario was predicted for Rossby waves on a parabolic zonal jet in Brunet and Warn (1990) when $|G| \ll 1$; Brunet and Haynes (1995) and Choboter et al. (2000) carried out a detailed study of the nonlinear evolution with wave breaking events. The invalidation of the linear expansion is only a necessary condition for a transition to a nonlinear regime; it is not sufficient. Section 4 will relate this criterion to the predisposition of the polar vortex to wave breaking processes in its interior.

Using results from Abramowitz and Stegun (1970, Eq. (13.5.1)) we obtain the following expression for the outer-region perturbation PV:

$$q(r,t) \propto r^{\mu/2}\left[\exp(-r^2(1+\mathrm{i}t)) + \frac{\Gamma((n/2)+(\mu/4)+1)}{\Gamma((n/2)-(\mu/4))}\frac{\mathrm{e}^{\mathrm{i}\pi(n+1)/2}}{(r^2 t)^{(\mu/2)+1}}\right] + \mathrm{O}(t^{-1}) \tag{2.28}$$

for $r^2 t \gg 1$. The first term in Eq. (2.28) is similar to the Brown and Stewartson (1980) expansion that is valid for monotonic wind profiles. Its radial gradient of perturbation PV increases linearly in time in the outer region as predicted from the palinstrophy law Eq. (C.2) of Appendix C. Note also that exact time-dependent energetic bounds are obtained in Appendix C using the palinstrophy law. The bounds show a maximum increase of kinetic energy when the radial gradient is largest. This kinetic amplification phenomenon in sheared flows is well known as Orr's (1907) effect (also see Thomson, 1887; Farrell, 1987).

The wave activity (WA) density q^*q of Eq. (2.28) (proportional to the angular pseudomomentum) has the following distribution for $r \sim \mathrm{O}(1)$

$$\mathrm{WA} = \frac{4}{\Gamma((Re(\mu)/2)+1)}(2r^2)^{Re(\mu)/2}\exp(-2r^2) \tag{2.29}$$

where for definiteness WA as been normalized to unity, i.e. $\int_0^\infty \mathrm{WA}r\,\mathrm{d}r = 1$. The mean radial position and the dispersion of the wave activity are

$$\bar{r} = \int_0^\infty \mathrm{WA}r^2\,\mathrm{d}r = \frac{Re(\mu)+2}{4}$$

and

$$D = \sqrt{\int_0^\infty \mathrm{WA}r^3\,\mathrm{d}r - \bar{r}^2} = \frac{Re(\mu)+2}{8} \tag{2.30}$$

respectively. The maximum of wave activity is attained in the outer region at $r_1 \approx (Re(\mu)/4)^{1/2}$. Note that the dispersion, the mean radius and maximum position as defined here are monotonically increasing functions of azimuthal wavenumber (see Eqs. (2.30) and (2.24)).

Now, the second term of Eq. (2.28) is proportional to the stationary solution of Eq. (2.10) which is proportional to $1/r^{(\mu/2+2)}$. Since the perturbation PV is a decreasing function of radius close to the origin and vanishes at the origin (see Eq. (2.25)) we expect this term (which is singular at the origin) to be responsible for a surge of wave activity when the inner region matches with the outer region. Note that this second term vanishes if $(2n-\mu)/4 = 0, -1, -2, \ldots$ and there is no reason in that case to expect an increase of wave activity in the inner region.

2.5. *Summary of linear analysis*

We have shown that three regions of wave activity with different dynamical properties develop in the outer, inner and matching regions for each azimuthal wavenumber in the linear approximation. The matching region is the region between the outer and inner regions and shrinks radially as $t^{-(1/2)}$. The outer region is asymptotically the recipient of most of the initial wave activity and the largest azimuthal scales tend to keep their wave activity closer to the origin. In the outer region, the perturbation enstrophy cascades to small radial scales and transfers all its kinetic energy to the zonal mean state. The inner region shows a wave activity decay varying as $|t^{-\sqrt{\gamma+n^2}}|$. The tendency for small azimuthal numbers to retain their wave activity is so efficient that the temporary increase in their radial gradients permits the amplitude of the nonlinear terms to catch up to the amplitude of the linear terms in the nonlinear PV equation when $4 > \gamma + n^2$. At this point, the Rossby wave propagation and shearing mechanisms are of the same order as the nonlinear self-advection terms and this could halt the perturbation enstrophy cascade for azimuthal wavenumbers smaller than $N = \sqrt{4-\gamma}$. This eventuality will be examined for the polar vortex in Montgomery and Brunet (2002) using linear and nonlinear numerical experiments. An equilibration between Rossby waves and nonlinear processes has been invoked in past studies (Rhines, 1975) for explaining the disruption of a cascade process, though recent work reminds us that this phenomenology is not without pitfalls (Legras et al., 1999).

3. Application to cyclone/anticyclone asymmetry

For our application to the cyclone/anticyclone asymmetry problem of the theory developed in Section 2, we use the exact solution Eq. (2.23) and its asymptotic form given by Eq. (2.25) to infer some general properties of a perturbed cyclonic and anticyclonic vortex monopole. Our definition of a cyclonic and anticyclonic vortex follows standard convention (see for example, Holton, 1992). To fix ideas, it is supposed that we are given a monopolar cyclonic and anticylonic vortex in gradient wind balance on the *f*-plane possessing identical distributions of tangential velocity with radius. The problem, then, is to determine the interior response of each vortex to a small-amplitude (linear) PV perturbation located just outside the core of the vortex. Analytical insight into this problem should help shed light on the observed dominance of anticyclonic vortices in 'freely-decaying' shallow water turbulence (Polvani et al., 1994; Arai and Yamagata, 1994).

The interior-region of a vortex monopole is represented by taking $\bar{\Omega}_0$ and $\bar{v}_0'''$ to be of opposite sign, and without loss of generality we assume $\bar{\Omega}_0$ is positive. Cyclone/anticyclone asymmetry may then be examined by simply changing the sign of f_0. A cyclonic vortex results with $f_0 > 0$, while an anticyclonic vortex results with $f_0 < 0$.

Eq. (2.23) governs the near-core disturbance PV and geopotential response to an exterior perturbation. As discussed in Section 2.4, the temporal decay of the disturbance PV and geopotential is controlled by $\mu = 2\sqrt{\gamma + n^2}$, where γ is defined by Eq. (2.9) and which can be written equivalently as

$$\gamma = 8 - 3!\frac{\bar{\Omega}_0}{\bar{v}_0''' L_{\mathrm{Ra}}^2}\left(1 + \frac{\bar{\Omega}_0}{f_0}\right)\left(1 + \frac{2\bar{\Omega}_0}{f_0}\right) \tag{3.1}$$

In Eq. (3.1), $L_{\mathrm{Ra}}^2 = \bar{\phi}_0/f_0^2$ denotes the square of an ambient Rossby deformation radius using the *ambient* Coriolis parameter. In the quasigeostrophic limit the mean vortex Rossby number $(\bar{\Omega}_0/f_0)$ is taken to zero:

$$\gamma_{\mathrm{QG}} = 8 - 3!\frac{\bar{\Omega}_0}{\bar{v}_0''' L_{\mathrm{Ra}}^2} \tag{3.2}$$

This illustrates the well-known 'symmetry' property of quasigeostrophic dynamics in that the decay of quasigeostrophic perturbations is independent of the sign of f_0.

Returning to the finite Rossby number case, let γ_{c} denote the normalized Rossby restoring term for the cyclonic vortex, and let γ_{ac} denote the corresponding term for the anticyclonic vortex. From Eq. (3.1) it follows that $\gamma_{\mathrm{c}} > \gamma_{\mathrm{ac}}$ and therefore $\mu_{\mathrm{c}} > \mu_{\mathrm{ac}}$. As a result, in the inner region (defined by $r^2 t = \mathrm{O}(1)$) the disturbance PV for the cyclonic vortex decays faster than the disturbance PV for the anticyclonic vortex. The nature of the inner region, however, implies that the cyclonic vortex will radiate vortex Rossby waves more quickly than the anticyclonic vortex. This together with Eq. (2.30) suggests that external perturbations (e.g. other vortices) will cause a cyclonic vortex to shed its core PV more rapidly and more extensively than the anticyclonic vortex, implying the anticyclonic vortex should be more resilient than the cyclonic vortex to forcing from the environment. Further work is required to understand and quantify the wave-mean flow interaction effects.

4. Application to tropical cyclone and polar vortex

To determine whether the theory developed in Section 2 is applicable to tropical cyclone vortices or the polar vortex one first needs to evaluate whether

$$\bar{\Omega} = \bar{\Omega}_0 + \bar{v}_0''' \frac{r^2}{3!} \tag{4.1}$$

represents a valid interior approximation of the angular velocity distribution of the basic state vortex. We recall from Section 2.1.1 that for this to be the case it is necessary that γ as defined by Eq. (2.5) be non-singular.

In tropical cyclones, the azimuthal mean vorticity distribution falls broadly into two classes: 'vortex monopoles', and 'vortex rings'. The monopole has its maximum relative vorticity at the center of circulation and is a useful idealization of the mean vorticity distribution in a developing tropical cyclone before attaining hurricane strength (e.g. Willoughby, 1990; Montgomery and Enagonio, 1998). The ring on the other hand, has its maximum relative vorticity at a finite radius (typically just inside the hurricane's eyewall) and is a useful idealization of the mean vorticity distribution of a mature tropical cyclone after attaining hurricane strength (Schubert et al., 1999). In the latter case, exponential or algebraic instabilities (not included here) will generally dominate the linear dynamics (Schubert et al., 1999; Smith and Rosenbluth, 1990; Nolan and Montgomery, 2000). We thus limit our focus to the former class. To the extent that a Gaussian vorticity distribution approximates observed azimuthal mean tropical storm vortices (Willoughby, 1990), we may safely assume that γ is nonsingular (as in Bassom and Gilbert, 1998). Taking characteristic parameter values for the vortices observed by Willoughby (1990), we find that $\mu > 4$. The local approximation and small amplitude expansion of Section 2 then appears well posed and uniformly valid for all time and for all azimuthal wavenumbers for a tropical cyclone vortex of this type. Explicit examples of the Rossby wave dynamics in the interior of tropical storm strength vortices (near or within the radius of maximum tangential velocity) and the relation to previous work are given in the companion paper of Montgomery and Brunet (2002).

The climatology of γ for the polar vortex does not show a singular behavior in at the pole, and moreover shows it to be approximately constant inside the vortex. Monthly means of the radial vorticity gradient for the polar vortex generally show positive values (see Fig. 2). The value of γ is then positive or negative depending on the magnitude of the curvature of the vortex's tangential wind profile. The *local approximation* employed herein thus permits a meaningful examination of the small-amplitude vorticity dynamics of the polar vortex interior. In the polar tangent-plane approximation, we obtain from Eq. (2.9)

$$\gamma = 8 - 6\,\frac{2(\Omega_{\mathrm{e}}/a^2) + (\Omega_0/L_{\mathrm{R}}^2)(\Omega_{\mathrm{e}} + \bar{\Omega}_0/2)(\Omega_{\mathrm{e}} + \bar{\Omega}_0)}{\bar{v}_0'''} \tag{4.2}$$

where Ω_{e} is the angular rotation rate of the earth. The interior of the polar vortex is defined here as the region interior to the tangential (zonal) velocity maximum. A typical upper stratospheric winter polar vortex has its monthly mean westerly jet maximum around latitude 40–50° latitude and the tropospheric westerly jet maximum centered around 30° latitude. Fig. 2 shows a 30-day time mean of the mean zonal wind for the period between 15 December 1994 to 14 January 1995 (Swinbank and O'Neill, 1994). In view of the remarks of Section 2.1

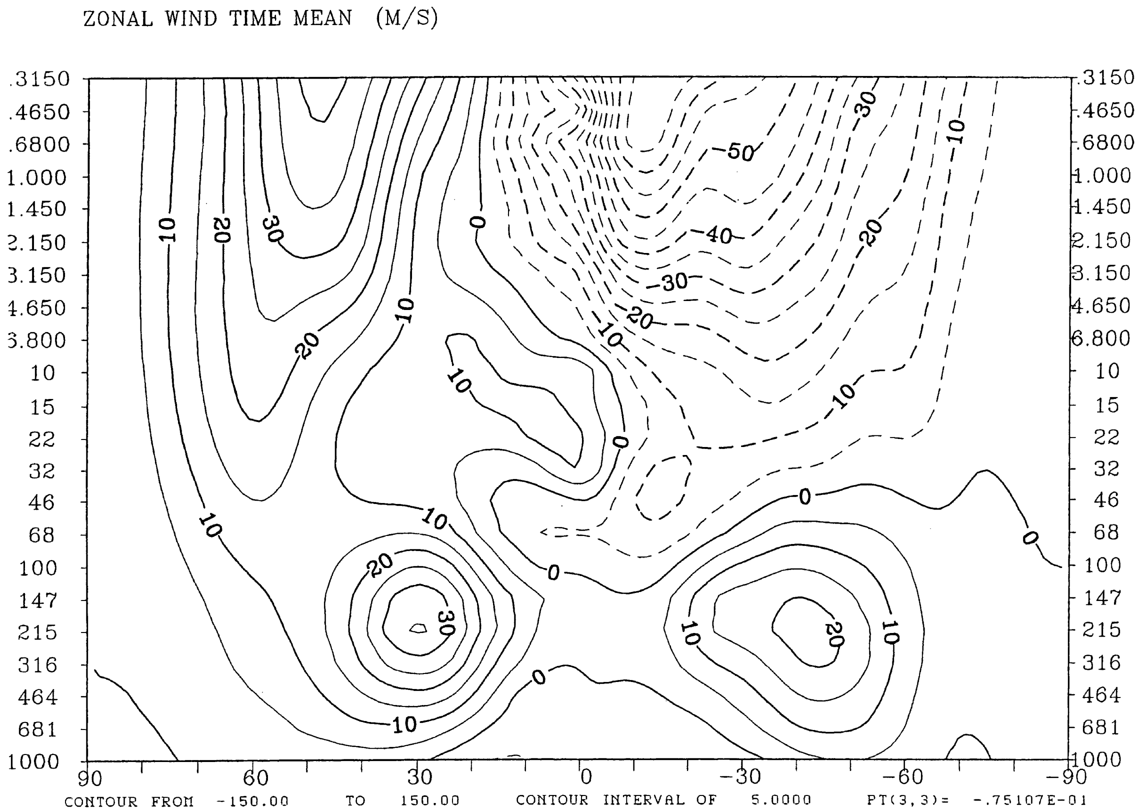

Fig. 2. Zonal and time mean of the zonal wind distribution for the period between 15 December 1995 and 14 January 1995. The abscissa is latitude and the ordinate is pressure in h Pa with a logarithmic scaling. Contour intervals are defined in 10 m/s intervals and negative values are dashed.

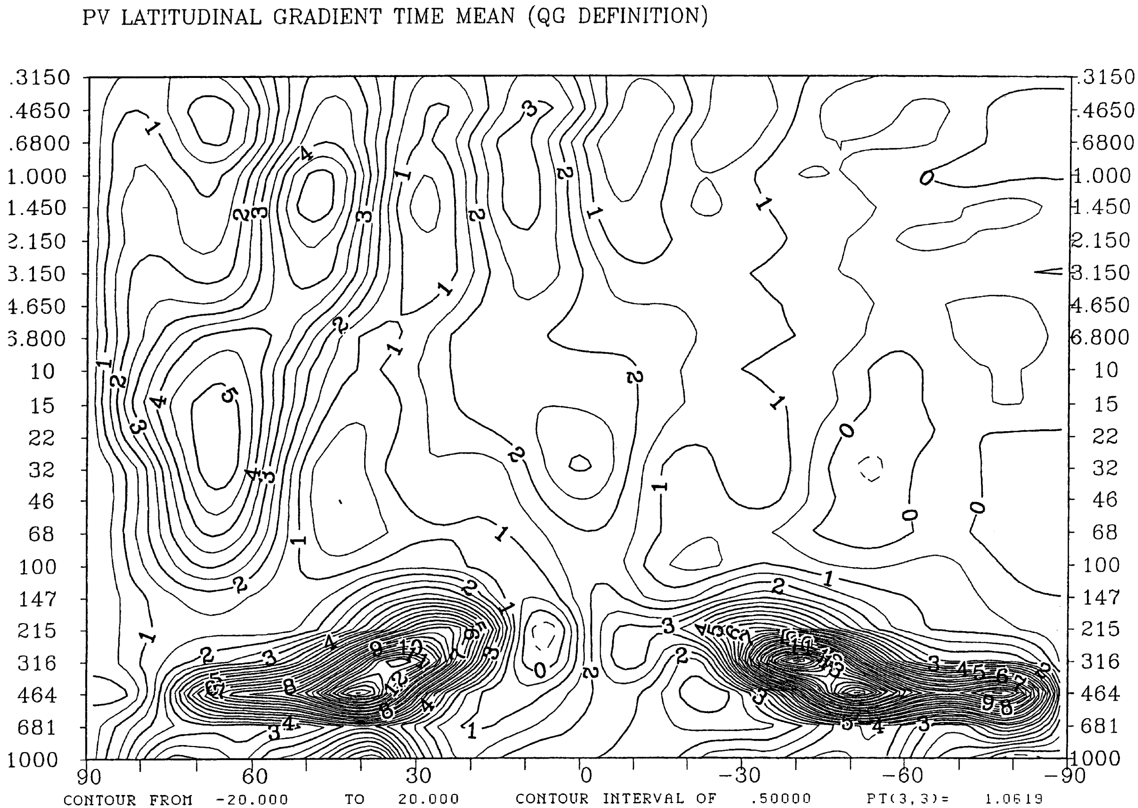

Fig. 3. Zonal and time mean of Ertel's LGPV for the period between 15 December 1995 and 14 January 1995. The LGPV has been normalized by the time mean density to make it consistent with the definition of quasigeostrophic theory. The abscissa is latitude and the ordinate is pressure in h Pa with a logarithmic scaling. Contour intervals are defined by the unit Ω_e/a.

regarding the polar tangent-plane approximation, Eq. (2.10) is a useful model for studying the barotropic dynamics of the interior of a polar vortex. The basic state angular velocity of the circumpolar vortex is then approximated by

$$\Omega_{\mathrm{F}} = \bar{\Omega}|_{r=0} + \tfrac{1}{2}\Omega_{\mathrm{e}}\Omega_2\varphi^2$$

where Ω_2 optimizes the RMS fit between Ω_{F} and the basic state angular speed. The fit is limited to the region defined by $\varphi_{\mathrm{i}} \geq \varphi \geq 0$, where φ_{i} is the colatitude (in radians) of the limit of the vortex interior. We have for a large radius of deformation

$$\gamma = 8 - \frac{4}{\Omega_2} \tag{4.3}$$

In the upper troposphere with $\varphi_{\mathrm{i}} = \pi/4$, we observe $\bar{\Omega}|_{r=0} \approx 2.6 \times 10^{-6}\,\mathrm{s}^{-1}$ with $\Omega_2 \approx 10^{-2}$ as typical values of the zonal monthly mean wind for winter. A crude estimate of γ using Eq. (4.3) then gives $\gamma \approx -4 \times 10^2$ for the upper troposphere. In the upper stratosphere, due to the very strong westerly jet, we obtain with the same methodology $\Omega_2 = 0.3$ and hence $\gamma \approx -4$.

To compare these γ values against the latitudinal gradient of potential vorticity (LGPV) as defined in quasigeostrophic theory, we transform γ to an equivalent LGPV by multiplying it by $-((\Omega_{\mathrm{e}}\Omega_2)/2a)$. This yields approximately $2(\Omega_{\mathrm{e}}/a)$ and $(2/3)(\Omega_{\mathrm{e}}/a)$ for the tropospheric and stratospheric values used above, respectively. Fig. 3 indicates that these estimates are consistent with the observed structure of Ertel's LGPV for the same period shown in Fig. 2. Note that in Fig. 3, Ertel's LGPV has been normalized by the time mean density to make it consistent with the definition of quasigeostrophic theory. The smaller stratospheric PV gradient shows how much more potent the radial shear can be in the wintertime stratospheric polar vortex compared to the upper-tropospheric jet (and other vortices, such as the interior region of a tropical storm as discussed previously). The large tropospheric γ indicates that the continuous-spectrum evolves quite slowly and that the shearing process is quite marginal in the upper tropospheric polar jet. The negative or near-zero values of γ in the wintertime stratosphere suggests on the basis of our simple model that shearing processes become important.

5. Conclusion

This paper has constructed a complete theory for linear vortex Rossby waves on a class of smooth circular vortices in a barotropic framework. The theory clarifies the nature and validity of the axisymmetrization process for small but finite amplitude potential vorticity disturbances excited near or interior to the tangential wind maximum of a vortex. For a given azimuthal wavenumber, a discrete expansion of vortex Rossby wave packets is unveiled whose radial structure is nonsingular and whose temporal decay is generally unlike that of simple sheared disturbances in rectilinear Couette flow.

The existence of a non-trivial discrete expansion of the continuous-spectrum is due to a space-time dilatation symmetry of the governing wave equation describing the shearing and Rossby mechanisms in the vicinity of the jet extremum. This insight into Rossby wave dynamics and their interaction with the mean vortex was also shown for rectilinear flow in the beta-plane approximation (Brunet, 1989), and hence is an indication of the robustness of the dynamics near a jet extremum.

Building on recent work of Montgomery and Kallenbach (1997), the wave mechanics developed for this class of vortices is believed a more fundamental route to understanding disturbed vortex interiors than the traditional approaches of Laplace transforms or integrals over singular eigenmodes. The theory points towards a deeper understanding of the vortex Rossby wave 'surf zone' for geophysical vortices constrained by rapid rotation and stable stratification in the large.

Several examples have been considered in order to demonstrate the usefulness of the theory.

1. The theory predicts that linear axisymmetrization in vortex interiors stays uniformly valid for all azimuthal wavenumbers for monopolar barotropic vortices on an *f*-plane. The nonlinear terms consequently stay small relative to the linear terms in the potential vorticity equation governing the balanced flow for small-amplitude disturbances. Given intermittent small-scale and small-amplitude vorticity forcing in the vortex interior, this implies an un-interrupted upscale energy cascade and vortex spin up.
2. The theory predicts that linear axisymmetrization in the interior of the polar vortex may be disrupted by the formation of nonlinear eddy structures for realistic polar-night configurations. The relatively weak latitudinal PV gradient and strong radial shear typical of the polar-night stratospheric vortex preconditions the interior of the vortex to Rossby wave breaking and Lagrangian stirring by the continuous-spectrum, like in Brunet and Haynes (1995). When the radial shear of the tangential flow becomes sufficiently weak the theory also points to the existence of azimuthally propagating discrete Rossby waves in the polar region, extending earlier work of Leblond (1964) which assumed a resting basic state.
3. The theory furnishes new insight into finite-depth effects beyond the quasigeostrophic approximation. For barotropic PV monopoles in gradient wind balance possessing a small but finite Rossby number and finite depth, the theory captures the leading-order cyclone/anticyclone asymmetry in the interior region of broadly distributed vortices. The theory predicts that for identical basic state tangential velocity distributions, perturbed anticyclonic vortices are more resilient than cyclonic vortices. This suggests the beginning of an explanation of anticyclonic dominance observed in inviscid barotropic free-decay scenarios (Polvani et al., 1994) via slow-manifold vortex Rossby wave mechanisms.

Extension of the present work along several fronts seems justified. Numerical exploration of the nonlinear breakdown of the linear expansion using the parameter space defined by Eq. (2.10) as a guide would be of great interest. This will be the subject of a companion paper (Montgomery and Brunet, 2002). The extension of the Schrödinger formulation to cover both the interior and exterior region of geophysical vortices would render the theory applicable to an even wider array of geophysical vortex problems, such as 'vortex alignment' following the formulation of Reasor and Montgomery (2001) and Schecter et al. (2002). A more comprehensive analysis of the cyclone/anticyclone asymmetry problem including an analysis of wave-mean-flow interaction and the effects of external disturbances on the vortex core is also warranted (McWilliams and Graves, private communication).

Finally, extension of the current theory to include coupling of a cyclonic vortex with the boundary layer and convection may help provide further understanding of the near-core spiral bands in tropical cyclones. Work along some of these lines is currently under way and the results will be reported in due course.

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Appendix A. Asymmetric balance

Asymmetric balance (AB) theory is a balance formulation for small amplitude asymmetric disturbances on a circular vortex in gradient wind balance (Shapiro and Montgomery, 1993; Montgomery and Franklin, 1998). The AB theory is a relative of the balance equations (McWilliams, 1985) and has been shown to give quantitatively correct results for freely axisymmetrizing disturbances on an intense vortex monopole with large-amplitude disturbances possessing many azimuthal wavenumbers (Moller and Montgomery, 1999). The theory possesses a complete set of pointwise conservation laws analogous to the linearized primitive equations, Charney-Stern and Fjortoft stability theorems, wave-activity conservation equations, and generalized stability theorems (Montgomery and Shapiro, 1995; Ren, 1999).

The mean vortex Rossby number is defined by $Ro = \bar{v}(r)/fr$. In the limit $Ro \to 0$, the AB theory reduces to quasigeostrophic theory, while for a finite depth and a small but finite Ro the AB theory possesses cyclone/anticyclone asymmetry. When considering the cyclone/anticyclone asymmetry problem, Ro will be assumed small but finite so that the AB approximation will be formally accurate for both cyclone and anticyclone evolution. The governing equation is given at the bottom of page 455 in Montgomery and Kallenbach (1997). Following Section 2.1.1, we consider near-core small-amplitude balanced dynamics on smooth monopolar vortices whose mean angular velocity distribution is given by

$$\bar{\Omega} = \bar{\Omega}_0 + \bar{v}_0''' \frac{r^2}{3!}$$

As in Section 2.1.1, the constant angular rotation rate in the azimuthal advection term is eliminated by defining a modified geopotential amplitude $\phi = \exp(-\mathrm{i}n\,\bar{\Omega}_0)\tilde{\phi}$. The differential equation then becomes isomorphic to Eq. (2.3) on invoking a 'vortex beta-plane' approximation in which the variable rotation rate of the core flow and the associated variation of the mean depth are neglected *except* where radially differentiated. The mathematical conditions are $r^2\bar{v}_0''' \ll 3\bar{\Omega}_0$ and $\bar{\phi}_0 \gg ((f_0\bar{\Omega}_0 + \bar{\Omega}_0^2)r^2)/2$, respectively, where $\bar{\phi}_0$ denotes the mean geopotential at $r = 0$. Applying these approximations to the single partial differential equation of Montgomery and Kallenbach (1997; pg. 455), Eq. (2.7) results. Note that the contribution to the mean radial radial PV gradient associated with the variation of the mean depth of the fluid is represented by the third term in (2.8).

Appendix B. Frequency-spectrum formula

After neglecting higher order terms in the expansion, the Taylor expansion of the potential well, see Eq. (2.16), around gives κ_{m}

$$V(\kappa) = \frac{\gamma}{(\kappa_{\mathrm{m}}^2 + 1)^2}\left[1 - \frac{4}{\kappa_{\mathrm{m}}^2 + 1}(\kappa - \kappa_{\mathrm{m}})^2\right] \tag{B.1}$$

where $\kappa_{\mathrm{m}} \geq \mathrm{O}(1)$. This is analogous to the harmonic oscillator treated by Cohen-Tannoudji et al. (1977, chapter 5) and the following law for the discrete spectrum holds:

$$\omega_{n,j} = \frac{\gamma}{(\kappa_{\mathrm{m}}^2 + 1)^2}\left[1 - 2(2j+1)\sqrt{\frac{\kappa_{\mathrm{m}}^2 + 1}{-\gamma}}\right]$$

where $j = 0, 1, 2, \ldots$. The normal mode for the fundamental frequency, that is $j = 0$, has a Gaussian distribution in momentum space

$$T_{n,j}(\kappa) = \exp\left(-\sqrt{\frac{-\gamma}{(\kappa_{\mathrm{m}}^2 + 1)^3}}(\kappa - \kappa_{\mathrm{m}})^2\right) \tag{B.2}$$

The case $\kappa_{\mathrm{m}} \ll 1$ is similar to the isotropic harmonic oscillator and is solved in Cohen-Tannoudji et al. (1977, chapter 7), where they obtain the following frequency formula:

$$\omega_{n,j} = \gamma\left(1 - 2\frac{(2j + n + 1)}{\sqrt{-\gamma}}\right) \tag{B.3}$$

where $j = 0, 1, 2, \ldots$.

Appendix C. Time-dependent energetic bounds

The following discussion is valid if the disturbance is initially and for all time square summable. Section 2.3 presented a complete solution to the initial-value problem, such that if an initial condition has a disturbance PV and geopotential in L^2, their evolution will stay in L^2 for all time when $G \geq 3/4$. To gain more insight into the dynamics of Eq. (2.15) with the potential (2.21) it is useful to discuss the different conservation laws available for this equation. A Lie group analysis was used by Brunet (1989) to obtain the following conservation laws, via Noether's theorem (Olver, 1986, chapter 4), for Eq. (2.15). The first two laws state that the total pseudomomentum, J, and total pseudoenergy, A, are conserved in time, where

$$J = -\frac{1}{2G}\int_0^\infty |T|^2\,\mathrm{d}\kappa \tag{C.1}$$

and

$$A = P + K$$

where

$$P = \frac{1}{2G}\int_0^\infty \left|\frac{\partial T}{\partial \kappa}\right|^2 \mathrm{d}\kappa \quad \text{and} \quad K = \frac{1}{2}\int_0^\infty \frac{|T|^2}{\kappa^2}\,\mathrm{d}\kappa$$

Physically, the conservation of angular pseudomomentum states that enstrophy is conserved, while the pseudoenergy law shows that kinetic energy K plus the potential energy P stored in the mean flow is conserved. Another conservation law particular to this wind profile gives the following algebraic relationship for the palinstrophy Π

$$\Pi = \int_0^\infty \kappa^2|T|^2\,\mathrm{d}\kappa = 8GA(t - t_{\mathrm{min}})^2 + \Pi_{\mathrm{min}}$$

where

$$t_{\mathrm{min}} = -\frac{1}{16GA}\left.\frac{\mathrm{d}\Pi}{\mathrm{d}t}\right|_{t=0} \tag{C.2}$$

and

$$\Pi_{\mathrm{min}} = \Pi|_{t=0} - \frac{1}{32GA}\left(\left.\frac{\mathrm{d}\Pi}{\mathrm{d}t}\right|_{t=0}\right)^2$$

Note that the conserved quantity is the sum of the constant terms on the RHS of Eq. (C.2). For G positive and $\mathrm{d}\Pi/\mathrm{d}t|_{t=0} \geq 0$ the palinstrophy increases monotonically in time, but for $\mathrm{d}\Pi/\mathrm{d}t|_{t=0} < 0$ it will attain a unique minimum before increasing monotonically. In general, the disturbances that generate the minimum are the ones with initial conditions that have lines of constant phase that tilt against the flow shear.

Using the Schwarz inequality, see Riesz and Sz-Nagy (1953), and Eq. (2.15) we can show the following inequality for the palinstrophy time tendency:

$$\left|\frac{\mathrm{d}\Pi}{\mathrm{d}t}\right|^2 \leq 32GP\Pi \tag{C.3}$$

The time evolution of Π, Eq. (C.2), and the inequality Eq. (C.3) can be used to derived the following bound for the minimum palinstrophy:

$$\Pi K|_{t=0}\frac{1}{A} \leq \Pi_{\mathrm{min}} \tag{C.4}$$

The minimum palinstrophy is positive definite for G positive.

The physical interpretation of the Eq. (C.2) is made easier if we normalize the vorticity, that is $J = -(1/2G)$, and we use the Schwarz inequality to obtain the following bounds on the kinetic energy:

$$\frac{1}{2\Pi} \leq K \leq \frac{\Pi_{\mathrm{min}}}{\Pi}A, \quad \forall G \geq 0 \tag{C.5}$$

K is bounded in its growth and decay for all initial conditions, but when the palinstrophy is minimum the lower and higher bound for the kinetic energy is the highest possible.

The palinstrophy attains this minimum when the vorticity develops its largest radial scales and it is then that kinetic energy can attain its largest value that is bounded by A. Asymptotically the kinetic energy will eventually decrease quadratically in time as shown by Eq. (C.5). The kinetic energy decay is due to wind shearing and is accompanied by the development of small radial scales as the palinstrophy increases quadratically in time.

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